Math 275D Lecture 12 Notes

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October 25, 2019

1 Polynomial Brownian Motion Martingales and Arcsine Laws

1.1 Polynomial Brownian motion martingales

What kind of function of Brownian motion is a martingale? We want $\mathbb{E}[f(t, B_t) \mid \mathcal{F}_s] = f(s, B_s)$ for all $t > s$. We can also state this as $\mathbb{E}[f(t, B(t)) - f(s, B(s)) \mid \mathcal{F}_s]$.

Proposition 1.1. *If f is a polynomial, and*

$$\frac{\partial f}{\partial t} + \frac{1}{2} f_{xx} = 0,$$

then $f(t, B_t)$ is a martingale.

Remark 1.1. This is not the heat equation, but it is similar. The heat equation looks like $\frac{\partial f}{\partial t} - \frac{1}{2} f_{xx} = 0$. If we let $p_t(x, y) = f_{B_t \mid \{B_0 = x\}}(y)$, then $p_t(0, x)$ satisfies the heat equation.

Remark 1.2. For high-dimensional Brownian motion, the formula should be

$$\frac{\partial f}{\partial t} + \frac{1}{2} \Delta f = 0.$$

How do we think of $p_t(x, y)$. Certainly, $\int p_t(0, y)\, dy = 1$. Here is how physicists think about it. If we have 1 pound of sand at $t = 0$, we can move the sand around randomly according to Brownian motion. Then at time $t = t_0$, $p_{t_0}(0, x)$ is the density of sand at x. The fact that $p_t(x, y)$ satisfies the heat equation explains why the variance of $p_t(0, y)$ spreads out as t grows (the probability spreads out like heat).

If f is a martingale, we get $\mathbb{E}[f(t, B_t) - f(0, B_0)] = 0$. What does this mean in physics? This is like

$$\int_{\text{sand}} f(t, Bt) = f(0, B_0).$$

Let $f_n = f(t, \text{position of sand particle } n)$. Then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(t) = f(0) = f(0, B_0).$$

What fs satisfy this condition? If f is constant or linear with respect to position, this condition holds. If you want a 2nd derivative condition, then you need $\frac{\partial f}{dt} + \Delta f = 0$.

Proof. We have $\mathbb{E}[f(t, B_t) - f(0, B_0)] = 0$. This is

$$\int f(t, y) p_t(x, y)\, dy - f(0, x) = 0.$$

If we take the derivative with respect to t, we get

$$\int f_1(t, y) p_t(x, y)\, dt + \int f(t, y) \frac{\partial p_t}{\partial t}(x, y) = 0.$$

By integration by parts, we get

$$\int (f_1 + \frac{1}{2} f_{2,2}) p_t(x, y) = 0.$$

So $f_1 + \frac{1}{2} f_{2,2} = 0$. $\square$

Remark 1.3. It is not necessary for f to be a polynomial. $f = e^{\theta B_t - \frac{1}{2}\theta^2 t}$ is also a martingale.

1.2 Arcsine laws and time of the maximum in Brownian motion

Let T be the first time such that $B_T = \sup_{t \in [0,1]} B_t$. Last time, we learned that the last zero of Brownian motion in $[0, 1]$ is distributed like arcsine. There are two other Brownian motion arcsine laws:

1. $|\{t : B(t) > 0, t \in [0, 1]\}|$,
2. T as defined above.

The way to calculate T is to first find the joint density of (T, M), where $M = \sup_{t \in [0,1]} B_t$. Let $M(t) = \sup_{s \in [0,t]} B_s$, and let $X_t = B(t) - M(t) \le 0$. We can also consider $Y_t = -|\tilde{B}(t)|$, which is a different Brownian motion. We claim that $X_t \stackrel{d}{=} Y_t$. Here is a heuristic argument

1. First, we have $|\{t : B(t) = M(t))\}| = 0$.
2. Next, if $B(t_0) \neq M(t_0)$, then there are an interval I and $t_0 \in I$ such that $B(t) - M(t)$ looks like a Brownian motion in I.
3. Now T for $B(t)$ is the last zero for $\tilde{B}(t)$. This is because the last zero of $\tilde{B}(t)$ and the last zero of Y_t have the same distribution. And Y_t and X_t have the same distribution.

The idea to prove this is to use a random walk. If we take a limit of scaled random walks, we will eventually get Brownian motion. We will go over this next time, in a result called Donsker's theorem.

If S_n is the result of a simple random walk on $\mathbb{Z}$ at time n, then let $X_n = S_n - M_n$. If $X_{n-1} \neq 0$, then $X_n = X_{n-1} \pm 1$ with probability $1/2$ each. If $X_{n-1} = 0$,

$$X_n = \begin{cases} -1 & \text{with probability } 1/2, \\ 0 & \text{with probability } 1/2. \end{cases}$$

Then $Y_n = -|\tilde{S}_n|$ has the same distribution as X_n. This result will extend to Brownian motion.